Anomalous heat equations based on non-Brownian descriptions

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HIGHLIGHTS

- Anomalous heat equations are proposed for non-Brownian heat conduction.
- Power-law length-dependence of the effective thermal conductivity are obtained.
- The length-dependence is connected to the mean-square displacement.
- The connections coincide with previous studies.

ARTICLE INFO

ABSTRACT

The Brownian description of heat conduction proposed by Razi-Naqvi and Waldenstrøm (2005) is generalized into non-Brownian anomalous heat conduction. It is found that there exists an entropic relation between the heat equation and Fokker–Planck equation (FPE) of energy fluctuations. Based on the entropic relation and fractional Brownian motion (FBM), we propose an anomalous heat equation (AHE), which is able to perform Brownian and non-Brownian long-time asymptotics of the mean-square displacement (MSD), $\langle \Delta \mathbf{x}^2 \rangle \propto t^\beta$. The AHE predicts a power-law length-dependence of the effective thermal conductivity κ_{eff} connected to the MSD, namely, $\kappa_{eff} \propto L^{\beta-1}$ with L denoting the system length. This scaling connection has been observed in the Lévy Walk (LW) model and linear response theory. Due to the coincidences with existing studies, the AHE can be considered as phenomenological models for anomalous heat conduction.

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1. Introduction

Heat conduction is generally described by Fourier's law, namely, $\mathbf{q} = -\kappa \nabla T$, where $\mathbf{q}(\mathbf{x}, t)$ is the heat flux, κ is the thermal conductivity and $T(\mathbf{x}, t)$ is the local temperature. In order to pose a closure problem, Fourier's law should be combined with the local energy conservation equation: $\partial_t e = -\nabla \cdot \mathbf{q}$ with $e = e(\mathbf{x}, t)$ the local energy density. Through the assumption $de = cdT$ with c denoting the specific heat capacity per unit volume, it can be reformed as $c\partial_t T = -\nabla \cdot \mathbf{q}$. In the case of constant κ and c, this combination leads to a parabolic heat conduction equation (PHCE), $\partial_t T = \alpha \nabla^2 T$ with $\alpha = \kappa/c$ standing for the thermal diffusivity. One well-known failure of Fourier's law is that the PHCE indicates infinite speeds of heat propagation [1]. According to Ref. [1], "the most obvious and simple generalization of Fourier's law that will give rise to finite speeds of propagation" is the Cattaneo's equation [2]: $\mathbf{q} + \tau \partial_t \mathbf{q} = -\kappa \nabla T$ with τ denoting the thermal relaxation time. Combining the Cattaneo's equation with the local energy conservation equation leads to a

hyperbolic heat conduction equation (HHCE) of T, namely, $\partial_t T + \tau \partial_t^2 T = \alpha \nabla^2 T$, which transmits heat waves with a finite speed $\sqrt{\alpha/\tau}$. The Cattaneo's equation also formulates the heat flux in terms of the integrated history of the temperature gradient,

$$\mathbf{q}\left(\mathbf{x}, t\right) = -\int_{-\infty}^{t} K\left(t - \xi\right) \nabla T\left(\mathbf{x}, \xi\right) d\xi, \, K\left(t\right) = \kappa \exp\left(-\frac{t}{\tau}\right). \tag{1}$$

Other constitutive models with memory behaviors will emerge from different choices of the memory kernel $K\left(t\right)$, which is required to be positive and converges monotonically to zero as $t \to +\infty$. For instance, the Jeffreys type [1] arises from the following kernel

$$K\left(t\right) = \kappa_F \delta\left(t\right) + \kappa_H \exp\left(-\frac{t}{\tau}\right), \tag{2}$$

wherein $\delta\left(t\right)$ is the Dirichlet function, κ_F is the thermal conductivity for Fourier heat conduction and κ_H is the thermal conductivity for hyperbolic heat conduction. Another typical example is $K\left(t\right) = \kappa_F \delta\left(t - \tau\right)$, which leads to the following single-phase-lagging model [3],

$$\mathbf{q}\left(\mathbf{x}, t + \tau\right) = -\kappa \nabla T\left(\mathbf{x}, t\right). \tag{3}$$

The single-phase-lagging model can be associated with the Cattaneo's equation through a first-order Taylor expansion [4]. If the memory kernel is chosen as the Mittag-Leffler function, a fractional-order constitutive equation will occur, which becomes the generalized Cattaneo equation (GCE) proposed by Compte and Metzler [5]. Besides memory (relaxation), there are other non-Fourier effects such as nonlocality and nonlinearity [6–22]. Nonlocality is usually observed in phonon heat transport, i.e., the phonon hydrodynamic [6,7] and Guyer–Krumhansl (GK) models [8,9], which is commonly reflected by second-order spatial derivatives like $\nabla^2 \mathbf{q}$ and $\nabla\left(\nabla \bullet \mathbf{q}\right)$. Nonlinearity can be found in the thermon gas theory [11–13], Lagrange multiplier model [20], hierarchy moment model [21], tempered diffusion model [22], and so on. The nonlinear heat conduction models give rise to an intriguing feature termed flux-limited behavior [15]. This behavior indicates that the heat flux will tend to a finite upper bound rather than infinity as the temperature gradient increases, which is sometimes paired with size-dependent existence [23].

Most of the above researches focus on the constitutive equations, whereas Razi-Naqvi and Waldenstrøm [24] have suggested a non-Fourier governing equation for the evolution of T. Their model was termed as "new heat equation (NHE)", which introduces a time-dependent effective thermal diffusivity in the heat conduction equation:

$$\partial_t T = \alpha\left(1 - e^{-t/\tau}\right) \nabla^2 T. \tag{4}$$

Obviously, the NHE can be regarded as a result of a time-dependent effective thermal conductivity κ_{eff}, namely,

$$\mathbf{q} = -\kappa_{eff} \nabla T, \, \kappa_{eff} = \alpha_{eff} c = \alpha\left(1 - e^{-t/\tau}\right) c. \tag{5}$$

Razi-Naqvi and Waldenstrøm have also compared the NHE with the PHCE, HHCE, equation of phonon radiative transfer (EPRT) [25] and ballistic–diffusive heat conduction equations (BDHCEs) [26]. The EPRT arises from a linearized Boltzmann transport equation (BTE), whose scattering term is approximated by the relaxation time. The EPRT is more physically meaningful than the HHCE but requires complicated calculations. The BDHCEs are also deduced from the BTE approach with the relaxation time approximation, which simplify the EPRT and are more accurate than the Cattaneo's equation. In this model, the phonon distribution function f is divided into two parts [26], $f = f_b + f_m$. f_b includes the heat carriers which originate from the boundaries and are not scattered, while f_m is for the rest of the heat carriers. The BDHCEs consist of the calculation for the ballistic heat flux $\mathbf{q}_b$ and the governing equation for the diffusive part of temperature T_m, which obeys the HHCE likewise. Compared with the Cattaneo's equation and BDHCEs, the NHE overcomes the artificial wave front in the diffusive regime [24], and its calculation is much more simplified than the EPRT.

In the long-time limit, the NHE reduces to normal heat diffusion, and the corresponding mean-square displacement (MSD) $\left\langle \Delta \mathbf{x}^2 \right\rangle$ should be Brownian, $\left\langle \Delta \mathbf{x}^2 \right\rangle \propto t$. Anomalous heat diffusion has nowadays attracted increasing interest [27–30], where the MSD obeys a non-Brownian growth, namely, $\left\langle \Delta \mathbf{x}^2 \right\rangle \propto t^\beta$. The range of β is usually classified into the following subranges: hyperdiffusion, $\beta > 2$, ballistic motion, $\beta = 2$, superdiffusion, $1 < \beta < 2$, normal diffusion, $\beta = 1$, and subdiffusion, $0 < \beta < 1$. The hyperdiffusion is a special subclass, which is impossible for initially thermalized particles [31]. In the following, we will show an entropic relation between the NHE and Fokker–Planck equation (FPE) of energy fluctuations. Based on the entropic relation, we propose several generalized heat conduction equations which can describe non-Brownian heat diffusion. These generalized models predict length-dependent κ_{eff} in anomalous heat diffusion, which agree with existing investigations. The main aim of our generalization is modeling anomalous heat conduction in low-dimensional systems, where the effective thermal conductivity commonly diverges to infinity in the long-time and/or large-size limit. Such divergence cannot be predicted by the NHE or other generalizations based on the EPRT because they will reduce to Fourier's law in the long-time and/or large-size limit. However, the universality of Fourier's law only exists for three-dimensional (3D) bulk materials. Consequently, the application scenarios of our study is fundamentally from that of previous generalizations. Divergence of the effective thermal conductivity means sharp enhancement of heat transport at the macroscale, which is intriguing in engineering.

2. Brownian and non-Brownian heat conduction

2.1. Heat conduction and energy fluctuations

As Razi-Naqvi and Waldenstrøm have mentioned, their approach is not new if $T = T(\mathbf{x}, t)$ is replaced by the probability density function (PDF) $P = P(\mathbf{x}, t)$, namely,

$$\partial_t P = \alpha \left(1 - e^{-t/\tau}\right) \nabla^2 P. \tag{6}$$

In the following, we will connect the PDF to energy fluctuations in heat conduction. In a paradigmatic review article [32], Dhar considered the correlation function of energy fluctuations $C_{ee} = C_{ee}(\mathbf{x}, t)$ in heat conduction, which is given by

$$C_{ee} = C_{ee}(\mathbf{x}, t) = \langle e(\mathbf{x}, t) e(\mathbf{0}, 0) \rangle - \langle e(\mathbf{x}, t) \rangle \langle e(\mathbf{0}, 0) \rangle. \tag{7}$$

Dhar assumed that C_{ee} fulfill the following diffusion equation:

$$\partial_t C_{ee} = \alpha_{eff} \nabla^2 C_{ee}. \tag{8}$$

In order to derive the Green–Kubo formula, α_{eff} should be constant, while we here allow it to be time-dependent. Then, one can acquire FPEs like Eq. (6) from Eq. (8) as if the PDF is defined as the normalization of C_{ee}, namely,

$$P(\mathbf{x}, t) = \left[\int C_{ee}(\mathbf{x}, 0) \, d\mathbf{x}\right]^{-1} C_{ee}(\mathbf{x}, t). \tag{9}$$

The PDF is also considered as the energy density profile [28,29], and the MSD is written as $\langle \Delta \mathbf{x}^2 \rangle = \int \left(\langle \mathbf{x}^2 \rangle - \langle \mathbf{x} \rangle^2\right) P d\mathbf{x}$, whose long-time asymptotics is determined by the FPE. A natural question arises: What is the relation between Eqs. (4) and (6)? We now address this question through entropy transport. Here, we employ the standard continuity equation, $\partial_t P = -\nabla \cdot \mathbf{J}$ with $\mathbf{J}$ is the probability current. This form is the most common continuity equation, which corresponds to the conservation law of the probability. There are also modified continuity equations for non-conserving systems, i.e., the fractional-order continuity equations [5]. As $\partial_t P = -\nabla \cdot \mathbf{J}$ is employed, one can find that Eq. (6) emerges from the combination of this continuity equation and a constitutive model $\mathbf{J} = -\alpha_{eff} \nabla P$. For constant α_{eff}, it describes Markovian normal diffusion, while time-dependent α_{eff} exhibits the non-Markovian character. The total entropy of the fluctuating system is written as $S = S(t) = -k_B \int P \ln P d\mathbf{x}$ with k_B denoting the Boltzmann constant. Thereupon, the local entropy density is given by $s = s(\mathbf{x}, t) = -k_B P \ln P$, whose time derivative reads

$$\begin{aligned}
\partial_t s &= -k_B (\ln P + 1) \partial_t P = k_B (\ln P + 1) \nabla \cdot \mathbf{J} \\
&= \nabla \cdot [k_B \mathbf{J} (\ln P + 1)] - k_B \mathbf{J} \cdot \nabla (\ln P).
\end{aligned} \tag{10}$$

In previous researches [33,34], $\int -k_B \mathbf{J} \cdot \nabla (\ln P) \, d\mathbf{x} = \int \alpha_{eff} k_B P^{-1} |\nabla P|^2 \, d\mathbf{x}$ is commonly regarded as the total entropy production rate induced by the system. Hence, the term $-k_B \mathbf{J} \cdot \nabla (\ln P)$ corresponds to the local entropy production rate σ. Owing to the local entropy balance equation $\partial_t s = -\nabla \cdot \mathbf{J}_S + \sigma$, the entropy flux $\mathbf{J}_S$ should satisfy

$$\mathbf{J}_S = -k_B \mathbf{J} (\ln P + 1) = \alpha_{eff} k_B (\ln P + 1) \nabla P = -\alpha_{eff} \nabla s. \tag{11}$$

Eq. (11) exhibits entropy transport driven by the entropy gradient. In the framework of classical irreversible thermodynamics (CIT) [35–37], the local entropy and entropy flux can be estimated by thermodynamic quantities, respectively,

$$s = \int c \frac{dT}{T}, \tag{12a}$$

$$\mathbf{J}_S = \frac{\mathbf{q}}{T}. \tag{12b}$$

Combining the above CIT formalism with Eq. (5) yields

$$\mathbf{q} = -\kappa_{eff} \nabla T \Leftrightarrow \mathbf{J}_S = -\alpha_{eff} \nabla s, \tag{13}$$

which coincides with Eq. (11) exactly. We now conclude a reasonable equivalence between non-Markovian energy fluctuations governed by Eq. (6) and non-Fourier heat conduction in Eq. (4), that they perform the same physical picture of entropy transport. Note that this equivalence only relies the CIT formalism. It is valid for arbitrary $\alpha_{eff} = \alpha_{eff}(t)$, which can deviate from $\alpha \left(1 - e^{-t/\tau}\right)$. In the nonlinear cases that the material properties are temperature-dependent, i.e., $\kappa = \kappa(T)$ and $c = c(T)$, Eq. (13) is still true. Though the PDF P is replaced by the temperature field T formally, their physical meanings are not interchangeable. The definition of P relies on the energy fluctuations, which will not exist as the fluctuations are ignored. However, T can always be defined whether the fluctuations are ignored or not. In the presence of non-negligible fluctuations, the connection between P and T can be presented through the entropy variation δs. In statistical mechanics, the entropy variation is given by $\delta s = -k_B (\ln P + 1) \delta P$, while for systems in (or near) local equilibrium, one can write $\delta e = T \delta s$ with δe the energy variation. Then, P and T are connected by the relation $(\ln P + 1) \delta P = -(k_B T)^{-1} \delta e$.

For heat conduction far from equilibrium, the conventional temperature in the sense of equilibrium or local equilibrium is not applicable either. Razi-Naqvi and Waldenstrøm therefore used Chen's [26] definition of the local temperature, which is defined as a measure of the local energy density: $e = cT$. However, the validity of $\partial_t e = c\partial_t T$ is controversial in the heat conduction far from equilibrium likewise. Several researchers [1] have suggested non-classical relations between the non-equilibrium temperature and energy density, which can give rise to non-Fourier governing equations as well. One conventional example takes the following form [38]

$$\partial_t e = c\partial_t T + \partial_t \left[\int_0^{+\infty} F(\xi) T(\mathbf{x}, t - \xi) d\xi \right],$$ (14)

where $F(\xi)$ is the so-called energy relaxation function. In the spirit of these studies, the NHE can also be considered as a result of the following continuity equation

$$c\partial_t T = \left(1 - e^{-t/\tau}\right) \partial_t e = - \left(1 - e^{-t/\tau}\right) \nabla \cdot \mathbf{q},$$ (15)

Meanwhile, the constitutive equation should remain Fourier's law. Eqs. (14) and (15) are proposed based on the same background but have no direct connection. We recall that the energy balance $\partial_t e = -\nabla \cdot \mathbf{q}$ still holds for both of them. The one-dimensional (1D) general form of the NHE is given by [24]

$$\partial_t T = \alpha_{eff}(t) \partial_x^2 T - b(t) \partial_x T.$$ (16)

In Eq. (16), $\alpha_{eff}(t)$ and $b(t)$ depends on the initial phonon injection at $x = 0$. Eq. (4) arises from the case that the initial injecting velocity obeys a Maxwellian distribution. Another special case is that the initial injecting velocity is parallel to the axis. In this case, $\alpha_{eff}(t)$ and $b(t)$ take the following forms [39]

$$\alpha_{eff}(t) = \alpha \left(1 - e^{-t/\tau}\right)^2, \, b(t) = v e^{-t/\tau}.$$ (17)

with v the magnitude of the injecting velocity. Eq. (17) is a straightforward application of Ref. [39] to heat conduction via proposing Brownian phonon transport [40]. Combining the standard continuity equation and Eq. (16) will yield a constitutive equation $q = cbT - \kappa_{eff} \partial_x T$. This constitutive equation predicts non-zero q even if $\partial_x T = 0$ and $|q|$ depends on the choice of coordinate axis for a given temperature field. The coexistence of $q \neq 0$ and $\partial_x T = 0$ is caused by the initial velocity at $x = 0$. Despite no driven force in the medium ($\partial_x T = 0$), the boundary injection can also induce non-zero heat flux. This constitutive equation also enables $q = 0$ to coexist with $\partial_x T \neq 0$, which leads to $\partial_t T = 0$ in the continuity equation. In Eq. (16), $\partial_t T = 0$ implies that the ratio $b(t)/\alpha_{eff}(t)$ is a time-independent constant. In general, the constitutive equation reduces to the Fourier's law in the limit $t \to +\infty$, which means $\lim_{t\to+\infty} \alpha_{eff}(t) = v^2\tau/3$ and $\lim_{t\to+\infty} b(t) = 0$. Then, we have $b(t)/\alpha_{eff}(t) \equiv 0$ and $b(t) \equiv 0$. Thus, the coexistence of $q = 0$ and $\partial_x T \neq 0$ requires that $b(t) \equiv 0$ and $\alpha_{eff}(t)$ has at least one zero point. The corresponding 1D entropy flux $J_S = q/T$ is given by

$$J_S = cb - \alpha_{eff} \frac{c}{T} \partial_x T = cb - \alpha_{eff} \partial_x s,$$ (18)

which deviates the form in Eq. (13). Besides the part induced by the entropy gradient, J_S will also contain an exponentially decaying part in the positive direction of coordinate axis. In the short-time limit, we have $\alpha_{eff}(t) = \alpha \left[2(t/\tau)^2 + o(t/\tau)^2 \right]$ and $b(t) = v[1 + o(t/\tau)]$, and thus $b(t)$ plays the dominant role when $t \ll \tau$. One possible inconsistency then occurs that q is in the same direction as $\partial_x T$ in some circumstances. For instance, when $\partial_x T$ is positive, q is also positive as if $t \ll \tau$. Positive $\partial_x T$ and q will be paired with negative CIT entropy production rate, which is calculated as $q\partial_x \left(\frac{1}{T}\right)$. It seems that the temperature gradient cannot exist in the positive direction of coordinate axis. Otherwise the Clausius statement of the second law of thermodynamics will be violated in the local volume element. Due to this inconsistency, we suggest that the constitutive equation remains Fourier's law. Combining Eq. (16) with Fourier's law will lead to a non-classical relation between e and T:

$$\partial_t e = c \frac{\alpha_{eff}}{\alpha} \partial_t T + cb\partial_x T.$$ (19)

2.2. Non-Brownian heat conduction

It is not difficult to find that Eq. (6) will be asymptotic to the standard force-free FPE as $t \to +\infty$. In the short-time limit, it becomes

$$\partial_t P = \alpha(t/\tau) \nabla^2 P, \quad t \ll \tau.$$ (20)

Note that for $t \ll \tau$, the system is far from equilibrium. Therefore, there may exist no well-defined local temperature and hence it is not appropriate to write $\partial_t T = \alpha(t/\tau) \nabla^2 T$. Interestingly, $\alpha_{eff} \propto t$ implies a ballistic growth of the MSD: $\langle \Delta \mathbf{x}^2 \rangle \propto t^2$. It indicates that the NHE corresponds to ballistic heat conduction in the short-time limit and will tend to be diffusive with increasing t. Furthermore, $\alpha_{eff} = \alpha \left(1 - e^{-t/\tau}\right)$ can be expanded as the following power series:

$$\alpha_{eff} = \alpha \left[\left(\frac{t}{\tau}\right) - \frac{1}{2!} \left(\frac{t}{\tau}\right)^2 + \frac{1}{3!} \left(\frac{t}{\tau}\right)^3 - \frac{1}{4!} \left(\frac{t}{\tau}\right)^4 + \cdots \right].$$ (21)

It implies that Eq. (4) can be considered as a linear superposition of heat diffusion with power-law time-dependent α_{eff}. The power-law time-dependence of α_{eff} has also been involved in anomalous diffusion termed fractional Brownian motion (FBM) [41], which is described in terms of the following FPE:

$$\partial_t P = \alpha \, (t/\tau)^{\beta-1} \, \nabla^2 P. \tag{22}$$

In the long-time limit, the MSD predicted by Eq. (22) is asymptotic to $\langle \Delta \mathbf{x}^2 \rangle \propto t^\beta$. The constitutive equation of Eq. (22) reads

$$\mathbf{J} = -\alpha \, (t/\tau)^{\beta-1} \, \nabla P, \tag{23a}$$

which also satisfies

$$\mathbf{J}_S = -\alpha \, (t/\tau)^{\beta-1} \, \nabla s. \tag{23b}$$

In the CIT formalism, Eq. (23b) yields

$$\mathbf{q} = -\kappa \, (t/\tau)^{\beta-1} \, \nabla T, \tag{24}$$

which gives rise to a governing equation with the same form as Eq. (22):

$$\partial_t T = \alpha \, (t/\tau)^{\beta-1} \, \nabla^2 T. \tag{25}$$

Eq. (25) emerges from FBM of energy fluctuations, which corresponds to the anomalous MSD. Accordingly, we shall term it as anomalous heat equation (AHE). The AHE exhibits fundamentally different behaviors from the NHE. The NHE crosses from ballistic heat conduction to diffusive heat conduction in the long-time limit, whereas the AHE is able to perform hyperdiffusive, ballistic, superdiffusive, normal and subdiffusive long-time asymptotics. Furthermore, the exponent β in the AHE is restricted to $1 \leq \beta \leq 2$ in the ballistic–diffusive heat conduction, whereas the power series expansion in Eq. (21) corresponds to $\beta \geq 2$. Consequently, the AHE can be regarded as a supplementary model for the NHE.

In the following, we will show that the AHE coincides with several previous studies on the effective thermal conductivity κ_{eff}. From Eq. (24), one can obtain a power-law time-dependence, namely, $\kappa_{eff} \propto t^{\beta-1}$. One typical treatment for the length-dependence is introducing the so-called "transit time" $t_T \approx L/v$ with L the system length, and we arrive at a power-law length-dependence, $\kappa_{eff} \propto t_T^{\beta-1} \propto L^{\beta-1}$. Interestingly, this connection between the κ_{eff} and β has been derived in previous investigations. Denisov et al. [27] acquired the same result in a 1D dynamical channel based on the model of Lévy Walk (LW). Without assuming specific random walk model, other researchers [28,29] recovered this relation through the Green–Kubo formula, which only needs the equilibrium autocorrelation function. Power-law length-dependent κ_{eff} have been widely observed in 1D or quasi-one-dimensional systems [32,42], i.e., the harmonic lattice, disordered harmonic lattice, and 1D momentum-conserving systems, while for two-dimensional (2D) cases, the length-dependence generally becomes logarithmic, $\kappa_{eff} \sim \ln L$. It should be mentioned that $\kappa_{eff} \propto L^{\beta-1}$ is not the unique connection between the length-dependence of κ_{eff} and MSD. For instance, Li and Wang [30] have deduced $\kappa_{eff} \propto L^{2-2/\beta}$ in 1D billiard gas channel models, which arises from the length-dependence of the mean first passage time (MFPT).

Another commonly used model for anomalous diffusion with the MSD $\langle \Delta \mathbf{x}^2 \rangle \propto t^\beta$ is the so-called fractal time process (FTP) [43], whose FPE takes the following form

$$\partial_t P = \alpha \tau^{1-\beta} D_t^{1-\beta} \nabla^2 P, \tag{26}$$

where $D_t^{1-\beta}$ is the Riemann–Liouville (RL) operator:

$$D_t^{1-\beta} \left[\nabla^2 P \, (\mathbf{x}, t) \right] = \frac{1}{\Gamma \, (\beta)} \frac{\partial}{\partial t} \int_0^t \frac{\nabla^2 P \, (\mathbf{x}, \varepsilon)}{|t - \varepsilon|^{1-\beta}} d\varepsilon. \tag{27}$$

When $P \, (\mathbf{x}, t)$ is replaced by $T \, (\mathbf{x}, t)$, we arrive at a fractional heat equation (FHE) [44,45]

$$\partial_t T = \alpha \tau^{1-\beta} D_t^{1-\beta} \nabla^2 T. \tag{28}$$

The fractional Fokker–Planck equation (FFPE) in Eq. (26) emerges from

$$\mathbf{J} = -\alpha \tau^{1-\beta} D_t^{1-\beta} \nabla P, \tag{29a}$$

while Eq. (28) corresponds to

$$\mathbf{q} = -\kappa \tau^{1-\beta} D_t^{1-\beta} \nabla T. \tag{29b}$$

Different from the integer-order cases stated above, Eqs. (29a) and (29b) are not equivalent in the framework of CIT. Upon series expansions, Eq. (29a) can be transformed into

$$\mathbf{J}_S = -\alpha \tau^{1-\beta} D_t^{1-\beta} \nabla s - k_B \alpha \tau^{1-\beta} \sum_{i=1}^{+\infty} \frac{(-1)^i}{i!} \frac{\Gamma \, (i + \beta - 1)}{\Gamma \, (\beta - 1)} \left[\partial_t^i \, (\ln P) \right] D_t^{1-\beta-i} \, (\nabla P), \tag{30a}$$

while Eq. (29b) corresponds to:

$$\mathbf{J}_s = -\alpha\tau^{1-\beta}D_t^{1-\beta}\nabla s + \kappa\tau^{1-\beta}\sum_{i=1}^{+\infty}\frac{(-1)^i}{i!}\frac{\Gamma(i+\beta-1)}{\Gamma(\beta-1)}\left[\partial_t^i\left(\frac{1}{T}\right)\right]D_t^{1-\beta}(\nabla T). \tag{30b}$$

It is shown that Eqs. (29a) and (29b) possess the same zero-order term, yet owing to the existence of the higher-order terms, they are different in entropy transport. A special case is stationary or quasi-stationary P and T. In this case, the higher-order terms in Eqs. (30a) and (30b) are zero or tend to zero. Then, Eqs. (29a) and (29b) are equivalent and we have

$$\mathbf{q}(\mathbf{x}, t) = -\frac{\kappa}{\Gamma(\beta)}\left(\frac{t}{\tau}\right)^{\beta-1}\nabla T(\mathbf{x}). \tag{31}$$

Eq. (31) means $\kappa_{eff} \propto t^{\beta-1}$, which agrees with the AHE. Even so, the FHE exhibits physical pictures fundamentally different from that of the AHE. In the AHE, the heat flux at time t is determined by the instantaneous temperature gradient at time t, whereas for the FHE, it depends on a integrated history of the temperature gradient in $[0, t]$.

3. Conclusions

1. Based on the frameworks of CIT and statistical mechanics, we establish an entropic connection between the NHE and FPE of energy fluctuations. In the FPE, the PDF is the defined as the normalization of the correlation function. It is shown that the two equations exhibit the same physical regime of entropy transport, that the entropy flux is driven the entropy gradient. A possible inconsistency is found in the general form of the NHE that the entropy production rate can be negative. This inconsistency can be avoided if a non-classical relation between e and T is introduced.

2. In the short-time limit, the NHE performs a ballistic MSD, while it tends to be diffusive with increasing t. Based on entropy transport in FBM, we propose the AHE, which is able to perform hyperdiffusive, ballistic, superdiffusive, normal and subdiffusive long-time asymptotics. The AHE can be regarded as a non-Brownian supplement for the NHE. The AHE predicts a length-dependence of the effective thermal conductivity connected to the MSD, namely, $\kappa_{eff} \propto L^{\beta-1}$. The same connection has been proved by existing theoretical results, which shows some reasonability of the AHE.

3. Anomalous diffusion from the fractional-order derivative, including the FFPE and FHE, is also studied. In contrast with the integer-order cases, entropy transport is no longer driven the entropy gradient, and due to the existence of higher-order terms, the two fractional-order equations are not equivalent. The FHE can predict the same time-dependence of κ_{eff} as the AHE, but their physical pictures are fundamentally different. The FHE implies a memory behavior between the heat flux and temperature gradient, which does not exist in the AHE.

4. As we have shown, the AHE can predict the scaling law between the divergent effective thermal conductivity and MSD, $\kappa_{eff} \propto t^{\beta-1}$. This scaling law has been proved by existing numerical studies in many low-dimensional systems, i.e., the hard-point particles model [46,47], single polymer chain of poly-3,4-ethylenedioxythiophene (PEDOT) [48] and Fermi–Pasta–Ulam (FPU)-β model [28,49]. In contrast, the NHE and its existing generalization [50] will reduce to Fourier's law in the long-time limit, which is paired with convergent effective thermal conductivity. Thus, the AHE will outperform existing approaches like the NTE and its existing generalization in low-dimensional situations.

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